Team Round Solutions

GLMT 2025

April 19, 2025

1. **[20]** In a right triangle, the smallest angle is π degrees. Find the largest angle in the triangle, in degrees.

   Proposed by: Muztaba Syed, Jacob Xu

   Solution. $\boxed{90}$

   The largest angle of a right triangle is always $\boxed{90}$ degrees. □

2. **[25]** A 12 by 20 rectangle is split into four congruent rectangles. Find the largest possible perimeter of one of the four rectangles.

   Proposed by: Muztaba Syed, Jacob Xu

   Solution. $\boxed{46}$

   It is optimal to keep the side length of 20. So each rectangle would be 3 by 20 and the perimeter would be $2(3+20) = \boxed{46}$. □

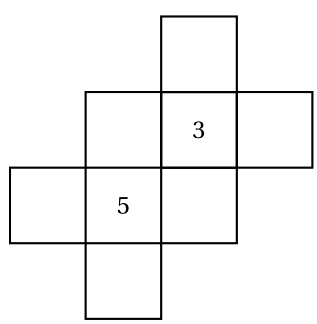
3. **[25]** There exists one non-degenerate triangle with integer side lengths such that the product of its sides equals 210. Find the perimeter of this triangle.

   Proposed by: Jacob Xu

   Solution. $\boxed{18}$

   The sides are constrained by the triangle inequality, so they must be close together. We see the sides must be 5, 6, 7, so the answer is $\boxed{18}$. □

4. **[30]** In the figure below, if a cell has more than one neighbor, its value is equal to sum of its neighbors. Find the sum of the values in the 6 empty cells.



   Proposed by: Muztaba Syed, Jacob Xu

   Solution. $\boxed{-8}$

   The two empty cells in the middle must both be 8. Then the sum of the two cells in the bottom left is $5-8-8=-11$, and the sum of the numbers in the top right is $3-8-8=-13$. The answer is then $-11+-13+8+8=\boxed{-8}$. □

5. **[30]** Suppose n is a positive integer such that if a fair n-sided die is rolled, the probability it rolls a square number is $\frac{1}{7}$. Find the sum of all possible values of n.

Proposed by: Jacob Xu

Solution. $\boxed{126}$

Since the probability given is $\frac{1}{7}$, n must be a multiple of 7. We can easily test multiples of 7 and notice that 35, 42, and 49 all work. So the answer is $\boxed{126}$.

The reason bigger numbers don't work is because after 49, when we increase the denominator of the probability by 7, the numerator either increases by 0 or 1. So all values starting from 56 fail. □

6. **[35]** Sylvia marks a point A in the Cartesian Plane. Lena starts at the lattice point closest to A, then travels in straight lines to the second closest, the third closest, and finally the fourth closest lattice points to A. Find the minimum possible length of the path Lena travels. Assume that no two of the four closest lattice points are equidistant to the point Sylvia selects.

Proposed by: Muztaba Syed

Solution. $\boxed{2+\sqrt{2}}$

Two lattice points are at least 1 unit apart, so the answer is at least 3.

Our first option is the path going from P to Q to R to S, where $PQRS$ is a square. But this is impossible since a point closest to P cannot be closer to R than S.

We can also go in a straight line. This fails since points off the line will be closer to A.

The next best thing we can do is $1+1+\sqrt{2}$. This can be achieved by putting the point at $\left(\frac{1}{3}, \frac{5}{8}\right)$. This gives a path from $(0,1)$ to $(1,1)$ to $(0,0)$ to $(1,0)$. □

7. **[35]** There are exactly two ordered pairs of real numbers (x, y) that satisfy $y = x^2 - 7x + 15$ and $x = y^2 - 7y + 15$. Find the sum of the values of x for these two ordered pairs.

Proposed by: Alexander Duncan

Solution. $\boxed{8}$

These equations are inverses, so if you reflect one equation across the line $y = x$ you will get the other equation. This means that if the graph of $y = x^2 - 7x + 15$ intersects the line $y = x$, then it will also intersect $x = y^2 - 7y + 15$ at the same point. To find the intersection of $y = x^2 - 7x + 15$ and $y = x$ we can substitute y for x to get $x = x^2 - 7x + 15$ so $x^2 - 8x + 15 = 0$ and $(x-3)(x-5) = 0$, giving the two ordered pairs $(3,3)$ and $(5,5)$. Thus, the answer is $3+5 = \boxed{8}$. □

8. **[40]** Find the sum of the digits of

$$9 \cdot (1 + 22 + 333 + 4444 + 55555 + 666666 + 7777777 + 88888888 + 999999999).$$

Proposed by: Muztaba Syed

Solution. $\boxed{54}$

Write each number as $\frac{10^k - 1}{9} \cdot k$, which when multiplied by 9 is just $10^k \cdot k - k$. When we sum this we get

$$9876543210 - (1 + 2 + \cdots + 9) = 9876543165.$$

This has sum of digits $\boxed{54}$. □

9. **[45]** Aurora, Bertha, Candice, and Denise each think of a distinct integer from 1 to 20, inclusive. They do not know what numbers the others are thinking of. They have the following perfectly logical conversation.

- Aurora: It is possible that my number divides all of your numbers.
- Bertha: My number could've as well, but now it definitely can't.

- Candice: My number is definitely a multiple of someone else's number. However, one of you may be thinking of a number bigger than mine.
- Denise: You are right, your number is a multiple of both Aurora and Bertha's numbers.
- Aurora: I now know everyone's numbers!

Find the product of their four numbers.

Proposed by: Muztaba Syed

Solution. $\boxed{720}$

The first dialogue tells us Aurora has 1, 2, 3, 4, or 5 (their number must have at least 3 multiples less than or equal to 20).

The second dialogue tells us that Bertha is also less than 6, but they can't be 1 or 2.

Candice can only know people divide them if they have 12 or 20. This is because of the numbers less than 6, only 5 does not divide 12, and only 3 does not divide 20. But their number can't be 20, since there is nothing larger than that.

Denise can tell that neither Aurora nor Bertha has 5, so they must have 5.

Finally Aurora knows that Bertha has 3 or 4 at this point. This means their number is one of 3 or 4 (it doesn't matter which).

Thus the answer is $3 \cdot 4 \cdot 12 \cdot 5 = \boxed{720}$. □

10. **[45]** Each term of the sequence

$$2, 3, 10, \ldots$$

is a product of the corresponding terms from two fixed arithmetic sequences. Find the 10th term of this sequence.

Proposed by: Isabella Li

Solution. $\boxed{227}$

The nth term of each of the two arithmetic sequences can be expressed as

$$a_1 + d_1 \cdot (n-1)$$

$$a_2 + d_2 \cdot (n-1)$$

And the product of these two is a quadratic involving n. Let $s_n = an^2 + bn + c$. We know that

$$s_1 = a + b + c = 2$$

$$s_2 = 4a + 2b + c = 3$$

$$s_3 = 9a + 3b + c = 10.$$

Solving, we get

$$a = 3, b = -8, c = 7.$$

So the answer is $s_{10} = 100 \cdot 3 - 10 \cdot 8 + 7 = \boxed{227}$. □

11. **[50]** Emmy puts the numbers 1 through 25 in the cells of a 5×5 grid. A cell is called *extreme* if it contains the smallest or largest number in its row or column. Find the maximum possible number of extreme cells.

Proposed by: Muztaba Syed

Solution. $\boxed{18}$

Note that 1 and 25 are always going to be the smallest and largest in their respective rows/columns. So there are at most 9 minimums and 9 maximums. We can also guarantee that no number is both a minimum and a maximum. The answer is then $\boxed{18}$.

It isn't obvious this works, but we can find the following construction by playing around with it. Here 1, 2, 3, 4, 6 are column minima; 1, 5, 7, 8, 9 are row minima; 20, 21, 23, 24, 25 are row maxima; 17, 18, 19, 22, 25 are column maxima.

1	2	3	4	20
5	10	11	21	6
7	12	13	14	23
8	18	15	16	24
17	9	19	22	25

Note: During the contest, some people interpreted the extreme condition as saying 1 or 25 are in the same row as the given cell. This gives an answer of $\boxed{16}$, which we also accepted. We apologize for the ambiguity. □

12. **[50]** Let x, y, and z be positive real numbers satisfying

$$(x+y)(x+z)=20,$$
$$(x+z)(z-x)=4, \text{ and}$$
$$(y-z)(y+z)=6.$$

Find $x+y+z$.

Proposed by: Selena Ge

Solution. $\boxed{\dfrac{15}{2}}$

Adding the first two equations gives $(z+x)(z+y)=24$. Adding all the equations gives $(y+x)(y+z)=30$. Multiplying these two new equations along with the first given equation gives $(x+y)^2(x+z)^2(y+z)^2=14400$. Thus, since x, y, and z are positive, $(x+y)(y+z)(z+x)=120$. Then, we can get that $x+y=5$, $y+z=6$, and $x+z=4$. Thus, adding these equations and dividing by 2 gives an answer of $\boxed{\dfrac{15}{2}}$. □

13. **[55]** Find the value of

$$\prod_{k=0}^{4}\left(2^{2^k}+\frac{1}{2^{2^k}}-1\right).$$

Proposed by: Isabella Li

Solution. $\boxed{\dfrac{2^{33}+2^{-31}+2}{7}}$

Observe the identity $\left(x+\frac{1}{x}+1\right)\left(x+\frac{1}{x}-1\right)=x^2+\frac{1}{x^2}+1$. Let the product be P. When we expand the terms, we have

$$P\left(x+\frac{1}{x}+1\right)=\left(x+\frac{1}{x}+1\right)\left(x+\frac{1}{x}-1\right)\left(x^2+\frac{1}{x^2}-1\right)\left(x^4+\frac{1}{x^4}-1\right)\left(x^8+\frac{1}{x^8}-1\right)\left(x^{16}+\frac{1}{x^{16}}-1\right).$$

Then by repeatedly using our identity this gives us $\left(x+\frac{1}{x}+1\right)P=x^{32}+\frac{1}{x^{32}}+1$. Thus $P=\boxed{\dfrac{2^{33}+2^{-31}+2}{7}}$. □

14. **[55]** In triangle ABC, let D, E, and F be the points where the incircle is tangent to sides BC, AC, and AB, respectively. Given that $AE=7$, $BF=8$, and $CD=9$, find $\cos(\angle FDE)\cdot\cos(\angle DEF)\cdot\cos(\angle EFD)$.

Proposed by: Selena Ge

Solution. $\boxed{\dfrac{21}{170}}$

Let $AD=a$, $BE=b$, and $CF=c$. By law of cosines, $\cos(A)=\frac{a^2+ab+ac-bc}{(a+b)(a+c)}$, and angle chasing gives $\angle FDE=\angle EFA=\frac{180-\angle A}{2}$. This means $\cos^2(\angle FDE)=\frac{bc}{(a+b)(a+c)}$. Thus, the quantity we want to calculate is $\frac{abc}{(a+b)(a+c)(b+c)}=\frac{21}{170}$. □

15. **[60]** In a 5×5 grid, Sasha draws paths between 4 distinct adjacent cells (not including diagonally adjacent), where the order of the cells matters. Find the number of paths Sasha draws.

Proposed by: Selena Ge

Solution. $\boxed{456}$

It helps to think of the paths as Tetris pieces to organize our casework.

- T pieces: There is no path which can reach all these cells, so this has no cases.
- S/Z pieces: There are 4 ways to orient these under reflection and rotation. Then there are 2 ways to determine the starting cell. Their location has $3 \cdot 4$ options, since they occupy a 2×3 rectangle. In total these contribute $4 \cdot 2 \cdot 12 = 96$ cases.
- L/J pieces: There are 8 ways to rotate/reflect these and two ways to choose the start. They also occupy a 2×3 rectangle, so there are $3 \cdot 4$ options for their location. In total this gives $8 \cdot 2 \cdot 12 = 192$ cases.
- O pieces: These can't be rotated, but there are 4 ways to choose the starting cell, and 2 ways to choose the direction of the path. They can be placed in 16 places, for a total of $4 \cdot 2 \cdot 16 = 128$ cases.
- I pieces: These can be rotated in 2 ways and have 2 possibilities for the starting cell. They can be placed in $2 \cdot 5$ places, for a total of $2 \cdot 2 \cdot 10 = 40$ cases.

In total our answer is $96 + 192 + 128 + 40 = \boxed{456}$ paths. □